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A CONSTITUTIVE MODEL FOR THE DEFORMATION INDUCED ANISOTROPIC PLASTIC FLOW OF METALS

Metals Behavior Branch
Metals and Ceramics Division

March 1979

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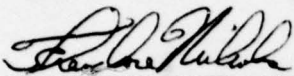79 06 14 044

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A mathematical framework is established for the equations governing inelastic deformation under multidimensional stress state and the associated evolution equations of the internal state variables. The formulation is based on a generalization of the Prandtl-Reuss flow law. For the evolution equations it is assumed that part of the rate of change in the inelastic state variables is isotropic and the remaining part varies according to the sign

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and orientation of the current rate of deformation vector. This leads to a minimum of 12 components of the internal state tensor to represent the inelastic plastic flow. In this manner, both initial and load history induced plastic anisotropy can be modeled. A specific set of equations for anisotropic plastic flow is developed consistent with the inelastic state variables.

FOREWORD

The work reported herein was performed at the Metals Behavior Branch, Metals and Ceramics Division, Air Force Materials Laboratory under Contract F33615-77-C-5003 with the Southeastern Center for Electrical Engineering Education. The work was conducted by Dr. Donald C. Stouffer, on leave from the University of Cincinnati and Dr. Sol R. Bodner, on leave from the Technion-Israel Institute of Technology in Haifa. Air Force administrative direction and technical support was provided by Dr. Theodore Nicholas, AFML/LLN.

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SECTION I

INTRODUCTION

An unresolved problem in the characterization of plastic deformation
of metals is the formulation of hardening laws that would realistically
represent hardening properties for completely general loading histories.
Most of the proposed theories follow classical plasticity so the problem
becomes one of predicting the size and shape of yield surfaces subsequent
to loading in the plastic range. These laws are generally some combination
of the isotropic and kinematic hardening models and reduce to either of
those as a special case. A feature common to many of these hardening
laws is a "back stress" term which corresponds to a translation of the
origin in the kinematic model. Examples of these hardening formulations
have been discussed recently in the proceedings of several conferences
(References 1, 2, 3), and others have since been proposed (References 4, 5).

For constitutive equations that do not require a yield criterion,
the corresponding problem is the determination of suitable evolutionary
equations for the internal state variables. These variables appear in
the equations for plastic deformation rate and represent the load history
dependent worked state of the material with respect to plastic flow.
Most of the earlier work in the development of such constitutive equations
employed one or two scalar functions of deformation history for the
inelastic state variables. As an example, Bodner and Partom (References
6, 7), used a single scalar function of plastic work as the inelastic
state variable for isotropic hardening conditions, e.g. uniaxial stress
with no Bauschinger effect. In this theory the plastic strain rate is
assumed to exist at all stress levels so the "back stress" concept does
not appear to be particularly useful since it would correspond to a non-
zero plastic deformation rate at zero applied stress.

The main problem in generalizing the constitutive equations to
multidimensional stress states is the development of suitable evolu-
tionary equations for the inelastic state variables. These equations
are required to account for the loading induced changes in resistance

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to plastic flow which vary in orientation as well as with the sign of the applied stress (or plastic deformation rate). The material therefore develops both orientational anisotropy and a multidimensional Baushinger effect which depends upon the complete loading history. Under the most general stress state with six stress components, a minimum of 12 inelastic state variables (components) would be required to characterize the material state with respect to plastic deformation; this assumes a single variable would apply for each orientation and direction of the plastic rate of deformation component.

The purpose of this report is to develop a proper mathematical formulation for the evolutionary equations required to represent the inelastic state of material subjected to general loading histories at large deformations. The procedure is motivated by earlier work of Bodner (Reference 8) and Onat (Reference 9) and the constitutive equations of Bodner and Partom (References 6, 7, 10, 11) but the development is completely general and could be applied to any set of constitutive equations that employ inelastic state variables. A consequence of this development is that a procedure is obtained that could account for induced plastic anisotropy for arbitrary loading histories. The limited comparisons possible with experimental data indicate that the proposed procedure is consistent with these observations.

SECTION II

A CONSTITUTIVE MODEL FOR ISOTROPIC DEFORMATIONS

Let d_{ij} represent the rate of deformation in an Eulerian coordinate system at any time t. The rate of deformation, defined as the symmetric part of the velocity gradient, is written as

$$d_{ij} = d_{ij}^e + d_{ij}^p \tag{1}$$

the composition of an elastic rate of deformation, d_{ij}^e, and an inelastic (or plastic) rate of deformation d_{ij}^p. The elastic component is fully recoverable whereas the inelastic component is nonreversible. The decomposition (Equation 1) is a very fundamental part of the current development since we assume that both components are non-zero under the action of all non-zero deviatoric stress components. A continuous representation of plastic flow is thereby obtained without the use of a yield surface or loading and unloading conditions.

This study is directed toward developing a representation for anisotropic plastic flow at large strains where the plastic anisotropy could be initial, deformation induced, or both. Elastic strains for metals are generally small even with large plastic deformations so it would be generally appropriate to use an isotropic elastic formulation if the material were initially isotropic. It is also well established that large plastic deformations have small effects on the elastic constants even with significant induced plastic anisotropy. The mathematical treatment of the elastic deformation rate d_{ij}^e at large strains would then be that given in Reference 11. Strong elastic anisotropy could be treated by classical linear anisotropic elasticity if the elastic strains were not large. The case of strong elastic anisotropy and large elastic strains is practically rare and would have to be treated as a special problem.

The governing law for isotropic, plastic deformation is taken in a form similar to the Prandtl-Reuss flow law

$$d^p_{ij} = \underline{d}^p_{ij} = \lambda(J_2, z_K, T)S_{ij} \tag{2}$$

The quantities $\underline{d}^p_{ij}$ and S_{ij} are the deviatoric components of the rate of plastic deformation and stress tensors. Thus Equation 2 implies that isotropic inelastic deformation is isochoric. The scalar material function λ is assumed to be a function of the second invariant of the deviatoric stress tensor, J_2, the absolute temperature, T, and a set of internal state variables, z_K. The quantities z_K are used to describe the state of the material microstructure at any time t; thus, they depend on the history of the deformation up to the current time t and are given by an evolution equation of the form

$$\dot{z}_K(t) = f_K(J_2, z_k, T) \tag{3}$$

To begin to establish a definite structure for Equations 2 and 3, let us use one specific component, z, of the vector z_K as a hardening term; e.g. an increase in z corresponds to an increase in the resistance to plastic flow. Thus assume the hardening rate, $\dot{z}$ is a monotonically increasing function of the plastic work

$$\omega_p(t) = \int_0^t S_{ij}(\tau)d^p_{ij}(\tau)d\tau \tag{4}$$

Note that this formulation neglects hardening recovery (softening). The evolution equation can be written as

$$\dot{z}(t) = g(\omega_p) \tag{5}$$

which can be expressed in the variables of Equation 3 as shown in Section V. After integrating, Equation 5 gives

$$z(t) = \overset{o}{z} + \int_0^t \dot{z}(\tau)d\tau \tag{6}$$

where $\overset{o}{z}$ is the initial hardness of the material.

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Under uniaxial flow conditions the material develops a directional characteristic so that the rate of deformation is different for loading in tension and compression; i.e., the Bauschinger effect. This effect can be expressed by introducing two hardness variables, $z^+(t)$ and $z^-(t)$ that are used for positive (tensile) and negative (compressive) rates of plastic deformation, respectively. Mathematically, then

$$z(t) = z^+(t) \; H \; (d_{11}^P) + z^-(t) \; H \; (-d_{11}^P) \tag{7}$$

where d_{11}^P is the uniaxial plastic deformation rate component and H is the Heaviside unit step function. Notice that using Equation 7 in Equation 4 does not produce a discontinuity in d_{11}^P since the jump from z^+ to z^-, or vis-a-vis, occurs when d_{11}^P is zero.

The values of $z^+(t)$ and $z^-(t)$ can be determined from Equation 6 once we develop a representation for $\dot{z}^+$ and $\dot{z}^-$, respectively. Recall $\dot{z}$ is defined as the total rate of hardening due to an axial flow d_{11}^P. Following the concepts of Reference 7 assume d_{11}^P is positive and let $q\dot{z}$ and $(1-q)\dot{z}$ correspond to the isotropic and directional hardening fractions* so that $\dot{z}^+(t) = q \; \dot{z} + (1-q)\dot{z} = \dot{z}(t)$ and $\dot{z}^-(t) = q \; \dot{z} - (1-q)\dot{z} = (2q-1)\dot{z}$. However if the d_{11}^P is negative rather than positive, then the hardening in tension and compression is given by $(2q-1)\dot{z}$ and $\dot{z}$, respectively. Thus we can write

$$\dot{z}^+(\tau) = q \; \dot{z} + (1-q)\dot{z} \left[\frac{d_{11}^P(\tau)}{|d_{11}^P(\tau)|} \right]$$

and

$$\dot{z}^-(\tau) = q \; \dot{z} - (1-q)\dot{z} \left[\frac{d_{11}^P(\tau)}{|d_{11}^P(\tau)|} \right] \tag{8}$$

*These are similar to isotropic and kinematic models of yield surface plasticity.

If q=1 then $\dot{z}^{+} = \dot{z}^{-}$ and the model corresponds to isotropic hardening; however, if q=0 then the model contains only directional hardening. If q<0 then the model predicts softening. Combining Equations 6, 7, and 8 gives

$$z(t) = \overset{o}{z} + \int_0^t q\, \dot{z}(\tau)\,d\tau +$$

$$\frac{d_{11}^P(t)}{|d_{11}^P(t)|} \int_0^t (1-q)\dot{z}(\tau)\, \frac{d_{11}^P(\tau)}{|d_{11}^P(\tau)|} \quad d\tau \tag{9}$$

The material parameter q could be approximated as a material constant but for more exact representations it may have to be treated as a history dependent internal state variable. The first integral represents the total isotropic hardening up to time t, whereas the total directional hardening used to characterize the Bauschinger effect is given by the second integral.

In the following Sections the above representations for isotropic and uniaxial plastic flow will be extended to a full three-dimensional anisotropic theory.

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SECTION III

CONSTITUTIVE FRAMEWORK FOR ANISOTROPIC PLASTIC FLOW

Consider the Prandtl-Reuss theory (Equation 2) for isotropic plastic flow. These laws have an important property; namely a given deformation rate component, d_{11}^p for example is parallel to the corresponding stress component S_{11}. We will show that this property carries over to all anisotropic materials.

Let us now rewrite the above theory to include anisotropic plastic flow. The most general anisotropic flow equation can be written as

$$d_{ij}^p = \hat{\lambda}_{ijk\ell} S_{k\ell} \tag{10}$$

where $\hat{\lambda}_{ijk\ell}$ is a fourth order tensor valued function of J_2, T, and state variables. Using the symmetry of d_{ij}^p and S_{ij} it follows that

$$\hat{\lambda}_{ijk\ell} = \hat{\lambda}_{jik\ell} = \hat{\lambda}_{ij\ell k} \tag{11}$$

Mathematically and conceptually it is convenient to replace the deviatoric stress and rate of deformation tensors in Equation 10 by six-dimensional vectors on the vector space $\hat{i}_\alpha$, $\alpha=1,2,\ldots 6$. Thus, let us define

$$\underset{\sim}{\hat{D}} = \hat{D}_\alpha \hat{i}_\alpha \quad \text{and} \quad \underset{\sim}{\hat{T}} = \hat{T}_\alpha \hat{i}_\alpha \tag{12}$$

and normalize the six-dimensional vector space such that

$$\sum_{\alpha=1}^{6} \hat{D}_i^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} (d_{ij}^p)^2 \quad \text{and} \quad \sum_{\alpha=1}^{6} \hat{T}_\alpha^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij}^2 \tag{13}$$

One such mapping, satisfying Equation 13 is *

$$\hat{D}_1 = d^P_{11} \qquad\qquad \hat{T}_1 = s_{11}$$

$$\hat{D}_2 = d^P_{22} \qquad\qquad \hat{T}_2 = s_{22}$$

$$\hat{D}_3 = d^P_{33} \qquad\qquad \hat{T}_3 = s_{33}$$

$$\hat{D}_4 = \sqrt{2}\ d^P_{12} \qquad\qquad \hat{T}_4 = \sqrt{2}\ s_{12}$$

$$\hat{D}_5 = \sqrt{2}\ d^P_{23} \qquad\qquad \hat{T}_5 = \sqrt{2}\ s_{23}$$

$$\hat{D}_6 = \sqrt{2}\ d^P_{31} \qquad\qquad \hat{T}_6 = \sqrt{2}\ s_{31}$$

$$\text{(14)}$$

Using this notation the plastic work, Equation 4 can be rewritten as

$$\omega_p = \int_0^t \hat{\underset{\sim}{D}}(\tau) \cdot \hat{\underset{\sim}{T}}(\tau)\ d\tau \tag{15}$$

The components of each of the two vectors $\hat{\underset{\sim}{T}}$ and $\hat{\underset{\sim}{D}}$ are linearly independent, as such the vectors $\hat{\underset{\sim}{T}}$ and $\hat{\underset{\sim}{D}}$ are orthogonal Cartesian vectors. If we let $\underset{\sim}{a}$ represent a three-dimensional proper orthogonal transformation such that $\underset{\sim}{S}' = \underset{\sim}{a}\ \underset{\sim}{S}\ \underset{\sim}{a}^T$, for example, then

$$\hat{\underset{\sim}{T}}' = \underset{\sim}{C}\ \hat{\underset{\sim}{T}} \quad\text{and}\quad \hat{\underset{\sim}{D}}' = \underset{\sim}{C}\ \hat{\underset{\sim}{D}} \tag{16}$$

where $\underset{\sim}{C}$ is a six-dimensional proper orthogonal transformation that can be easily constructed from the components of $\underset{\sim}{a}$.

Let us now rewrite Equation 10 on the six-dimension of vector space as

$$\hat{\underset{\sim}{D}} = \hat{\underset{\sim}{\lambda}}\ \hat{\underset{\sim}{T}} \quad\text{or}\quad \hat{D}_\alpha = \hat{\lambda}_{\alpha\beta}\hat{T}_\beta \tag{17}$$

---

*A mapping similar to this, except on a five-dimensional space, is given by Ilyushin and Lensky (Reference 12).

where $\hat{\underset{\sim}{\lambda}}$ is a six-dimensional square tensor. Let us assume that $\hat{\underset{\sim}{T}}$ can be determined from a potential function. Then using commutativity of the partial derivative, it follows that

$$\hat{\underset{\sim}{\lambda}} = \hat{\underset{\sim}{\lambda}}^{T} \quad \text{or} \quad \hat{\lambda}_{\alpha\beta} = \hat{\lambda}_{\beta\alpha} \tag{18}$$

Since $\hat{\underset{\sim}{\lambda}}$ is symmetric and the components are real, there exist real eigenvalues, $\lambda_{\alpha\alpha}$. Thus using $\underset{\sim}{\lambda} = \underset{\sim}{Q} \hat{\underset{\sim}{\lambda}} \underset{\sim}{Q}^{T}$, where $\underset{\sim}{Q}$ is a proper orthogonal transformation, Equation 17 becomes

$$\underset{\sim}{D} = \underset{\sim}{\lambda} \underset{\sim}{T} \tag{19}$$

where $\underset{\sim}{\lambda}$ is a 6 x 6 diagonal tensor. The rate of deformation, $\underset{\sim}{D}$, and stress, $\underset{\sim}{T}$, are vectors that use the eigenvectors $\hat{\underset{\sim}{e}}_{\alpha}$ of $\underset{\sim}{\lambda}$ as a basis. Equation 19 has the advantage that all the anisotropy properties can be described by the six eigenvalues $\lambda_{\alpha\alpha}$ and the transformation $\underset{\sim}{Q}$ so that the structure of the equation is particularly simple. The details for determining $\underset{\sim}{Q}$ are delayed until the end of the next section.

Next we assume that each non-zero component of $\underset{\sim}{\lambda}$, can be expressed as a single valued scalar function of the scalars J_2, T, and $Z_{\alpha\beta}$, where $Z_{\alpha\beta} = 0$ for $\alpha \neq \beta$, so that each non-zero component $Z_{\alpha\beta}$ corresponds directly to a non-zero component of $\lambda_{\alpha\beta}$. Mathematically, this is written as

$$\lambda_{\alpha\alpha} = f(J_2, T, Z_{\alpha\alpha}), \quad \text{no sum on} \quad \alpha \tag{20}$$

Since $\lambda_{\alpha\alpha}$ are diagonal elements of a second order tensor in the $\hat{\underset{\sim}{e}}_{\alpha}$ basis, then Equation 20 implies that $Z_{\alpha\alpha}$ must also be diagonal components of a second order tensor on the basis $\hat{\underset{\sim}{e}}_{\alpha}$, thus

$$\underset{\sim}{Z}' = \underset{\sim}{C} \underset{\sim}{Z} \underset{\sim}{C}^{T} \tag{21}$$

In the isotropic theory relating to Equation 2 we imposed the condition that the plastic flow is isochoric. For anisotropic flows this

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condition leads to considerable mathematical difficulties as well as a
physically inconsistent theory. Using Equation 19 consider the sum

$$D_1 + D_2 + D_3 = \lambda_{11} T_1 + \lambda_{22} T_2 + \lambda_{33} T_3 \tag{22}$$

If the material is isotropic, then $\lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda$ and Equation 22
becomes

$$D_1 + D_2 + D_3 = \lambda(S_{11} + S_{22} + S_{33}) = 0 \tag{23}$$

for every choice of λ. Thus the anisotropic flow model will reduce to
classical incompressible isotropic plasticity. However the volumetric
change $D_1 + D_2 + D_3$ will not be identically zero for anisotropic flows.
This appears to be consistent with experimental observations, since the
plastic Poisson ratio measured during plastic deformation is generally
different from one half (Reference 15).

SECTION IV
STRUCTURE OF THE STATE VARIABLES

To establish a representation of Z for anisotropic plastic flow we follow the concepts underlying Equation 9. This was used to introduce a nonsymmetric property into the plastic flow equation so that the response in tension and compression, or shear to the left and right, are different. To maintain this characteristic let us define $Z_{\alpha\alpha}$ in the following manner

$$Z_{\alpha\alpha}(t) = Z_{\alpha\alpha}^{+}(t)H[D_{\alpha}(t)] + Z_{\alpha\alpha}^{-}(t)H[-D_{\alpha}(t)] \qquad (24)$$

where $Z_{\alpha\alpha}^{+}(t)$ and $Z_{\alpha\alpha}^{-}(t)$ are the hardness variables for positive and negative plastic deformation rates.

It is informative to consider three different classes of mechanical properties.

a) A material is defined as <u>orientationally isotropic</u> if

$$Z_{\alpha\beta}^{+}(t) = z^{+}(t)\ I_{\alpha\beta} \qquad (25a)$$

and

$$Z_{\alpha\beta}^{-}(t) = z^{-}(t)\ I_{\alpha\beta} \qquad (25b)$$

for all time and deformation histories. The quantity $I_{\alpha\beta}$ is a six-dimensional identity tensor,

$$I_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \qquad (26)$$

Thus, according to Equations 25 and 23 for equal values of stress, T_{α}, the deformation rates are the same in all six coordinate directions of the same sign. Further using either Equation 25a or 25b in Equation 21 gives

$$(z^{+})' = C(z^{+}I)C^{T} = z^{+}I \qquad (27)$$

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which shows that $\underset{\sim}{Z}^+$ (and $\underset{\sim}{Z}^-$) are identical in all coordinate directions. Note that this result follows since $\underset{\sim}{Z}$ transforms as a second order diagonal tensor.

b) A material is skew symmetric if

$$Z^+_{\alpha\beta}(t) = Z^-_{\alpha\beta}(t) \tag{28}$$

for all time and deformation histories. This corresponds to no Bauschinger effects in a uniaxial deformation history. Using Equations 19, 20, and 28 we can write, for example,

$$f(J_2, T, Z^-_{\alpha\alpha}) (-T_{\alpha\alpha}) = -f(J_2, T, Z^+_{\alpha\alpha})T_{\alpha\alpha}$$

which is skew symmetry of the material function f.

c) A material is defined as isotropic if it is both orientationally isotropic and skew-symmetric.

Consider now the internal state tensor $\underset{\sim}{Z}$. Since the only non-zero components of $\underset{\sim}{Z}$ are on the diagonal, $Z_{\alpha\alpha}$ are eigenvalues of $\underset{\sim}{Z}$ and $\hat{e}_\alpha$ are the eigenvectors. Thus $\underset{\sim}{Z}$ can be written as a six-dimensional vector in the form

$$\underset{\sim}{Z}(t) = Z_{\alpha\alpha}(t) \, \hat{e}_\alpha \tag{29}$$

Referring to the structure of Equation 9, let us rewrite the vector $\underset{\sim}{Z}$ in the same general form,

$$\underset{\sim}{Z}(t) = \overset{o}{\underset{\sim}{Z}} + \int_0^t \dot{\underset{\sim}{Z}}^I(\tau)d\tau + \int_0^t \dot{\underset{\sim}{Z}}^A(\tau)d\tau \tag{30}$$

The vector $\overset{o}{\underset{\sim}{Z}}$ is used to specify the initial hardness in each of the six coordinate directions. The vectors $\dot{\underset{\sim}{Z}}^I$ and $\dot{\underset{\sim}{Z}}^A$ correspond to the rate of fully isotropic and directional hardening, respectively. Note that isotropic hardening is characterized by the first integral in Equation 9 whereas the second integral in Equation 9 characterizes orientationally

anisotropic and non-skew symmetric hardening. Thus we can use the first integral from Equation 9 directly into Equation 30 to obtain

$$\underset{\sim}{Z}(t) = \overset{o}{\underset{\sim}{Z}} + \hat{e}_\alpha \int_0^t q \, \dot{z}(\tau) d\tau \; + \; \int_0^t \overset{\cdot A}{\underset{\sim}{Z}}(\tau) \, d\tau \tag{31}$$

where $\dot{z}(\tau)$ is calculated from Equation 5 using the total work as given by Equation 15. Recall q is a material parameter describing the relative amount of fully isotropic hardening.

Using Equation 19 it can be seen that there is a one to one relationship between a deviatoric stress component T_α and the corresponding rate of deformation D_α. However, it is not expected that the vectors $\underset{\sim}{T}$ and $\underset{\sim}{D}$ are parallel since the components of $\underset{\sim}{Z}$, and hence λ, are not necessarily equal. Let us assume that the rate of change of resistance to plastic flow (hardness) is due essentially to the plastic straining. That is, the change in resistance to plastic flow is in the direction of the rate of plastic deformation $\underset{\sim}{D}$; so that $\overset{\cdot A}{\underset{\sim}{Z}}$ and $\underset{\sim}{D}$ are parallel. Expressed mathematically

$$\overset{\cdot A}{\underset{\sim}{Z}} = \left| \overset{\cdot A}{\underset{\sim}{Z}} \right| u_\alpha \, \hat{e}_\alpha \tag{32}$$

where u_α are the "direction cosines" of the rate of deformation vector. Hence we can write

$$u_\alpha = \frac{D_\alpha}{|\underset{\sim}{D}|} \tag{33}$$

and $|\underset{\sim}{D}| = (\underset{\sim}{D} \cdot \underset{\sim}{D})^{1/2}$. This formulation gives the hardness as varying according a cosine rule as proposed by Bodner (Reference 8). Using again the second integral of Equation 9 and noting that $\left| \overset{\cdot A}{\underset{\sim}{Z}} \right| = (1-q)\dot{z}(\tau)$ we can write

$$\underset{\sim}{Z}(t) = \overset{o}{\underset{\sim}{Z}} + \hat{e}_\alpha \int_0^t q \, \dot{z}(\tau) \, d\tau \; +$$

$$\hat{e}_\alpha \frac{D_\alpha(t)}{|D_\alpha(t)|} \int_0^t (1-q)\dot{z}(\tau) \, u_\alpha(\tau) d\tau \tag{34}$$

Thus we have a specific form for the evolution of the tensor Z. This representation is based on the main assumption that the rate of anisotropic hardening vector is parallel to the rate of plastic deformation vector. The total anisotropy at any time depends on the integral of $\dot{Z}(\tau)$ or the history of the deformation. Thus, the representation has the necessary path dependence.

Returning to Equation 19, it is necessary to know the transformation Q in order to obtain a diagonal form for the material tensor $\hat{\lambda}$. From Equation 34 it can be seen that Z is written relative to the basis e_α, the eigenvectors, which are fixed during the deformation history $(0,t)$. Thus the choice of Q depends only upon the initial anisotropy $\overset{o}{Z}$, which is assumed to be a known initial condition. Further if the material is initially isotropic, then every set of vectors $\hat{e}_\alpha$ are eigenvectors and the vectors D and T can be chosen such that the components are physically convenient for the shape of the body.

There is a limited amount of experimental evidence available for the anisotropic deformation of beryllium. In general, Nicholas (Reference 13) has shown that a comprehensive anisotropic hardening theory is necessary to predict the experimental results for beryllium. The results published by Lindholm, Yeakly, and Davidson (Reference 14) for beryllium in biaxial tension showed that the above theory is qualitatively consistent with the data in that the stress-strain curves in the plastic range are ordered with loading history according to the cosine law given by Equation 32.

SECTION V

A SPECIFIC REPRESENTATION

The purpose of this Section is to briefly summarize the governing equations for deformation induced anisotropic plastic flow in a form consistent with the Bodner-Partom model for isotropic plastic flows. Squaring Equation 2 gives

$$D_2^p = \lambda^2(J_2, z_k, T)J_2 \tag{35}$$

where D_2^p and J_2 are the second invariants of d_{ij}^p and S_{ij}, respectively. Substituting Equation 35 into Equation 2 gives

$$d_{ij}^p = \sqrt{D_2^p(J_2, z_k, T)} \ \frac{S_{ij}}{\sqrt{J_2}} \tag{36}$$

This result is reminiscent of the Prandtl-Reuss theory except that no yield surface is used with Equation 36; i.e., D_2^p is continuous. Using one state variable and motivated by the equations of dislocation dynamics, Bodner and Partom selected D_2^p such that

$$d_{ij}^p = D_0 \exp\left[-\tfrac{1}{2}\left(\frac{z^2}{J_2}\right)^n\right] \ \frac{S_{ij}}{\sqrt{J_2}} \tag{37}$$

where the constant D_0 is the limiting strain rate of the material and n, generally a function of temperature, relates to the strain rate sensitivity. Since z is a single internal variable that controls the rate process it clearly must be a measure of the resistance to plastic flow of the material. Specifically Equation 5, is chosen in the form

$$\dot{z}(\tau) = m(z_1 - z_0) \exp[-m\,\omega_p(\tau)]\dot{\omega}_p = m(z_1 - z)S_{ij}\,d_{ij}^p \tag{38}$$

where m, z_1, and z_0 are constants controlling the rate and level of the hardening characteristics, Equation 38 is consistent with the general form for $\dot{z}$, Equation 3.

The representation for anisotropic flows can be completely determined by combining the above isotropic representation with the results in Sections III and IV. The plastic deformation rate, Equation 19 becomes

$$D_\alpha = D_0 \exp\left[-\tfrac{1}{2}\left(\frac{z_{\alpha\alpha}^2}{J_2}\right)^n \right] \frac{T_\alpha}{\sqrt{J_2}} \qquad (39)$$

where D_α and T_α are the six-dimensional rate of deformation and deviatoric stress vectors defined by the mapping (Equation 14). The components of the hardening tensor $Z_{\alpha\alpha}$ are obtained by substituting Equation 38 into Equation 34.

The anisotropic plastic representation, Equations 39, 38, and 34, requires the use of five material constants (q, n, m, z_1, z_0) that must be determined experimentally (D_0 is a scale factor). It is useful to note that these material parameters can all be determined from a one-dimensional experimental program. There were no material functions introduced by the anisotropy development. This leads to a specific model that can be readily evaluated for different materials.

In closing it is important to note that the constitutive equations are a nonlinear coupled system that can, most likely be solved numerically. Numerical finite difference and finite element procedures have been developed to solve boundary value problems under isotropic hardening conditions. It is expected that these codes can be modified to include the anisotropic model; however, to date no numerical work has been done using the above system of equations.

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